Tutorial 4. Oct. 05. 2015

~~1. Evaluate the integrals.~~

1. (6) Differentiate the following functions using definition.

(i) $f(z) = z^2 + z$, (ii) $f(z) = 1/z$ $(z \neq 0)$; (iii) $f(z) = z^3 - z^2$

2. (5) (i) In each of the following cases, write $f(z)$ in the form $u(x,y) + iv(x,y)$ where $z = x + iy$ and u, v are real-valued functions.

(a) $f(z) = z^2$. (b) $f(z) = \frac{1}{z}$ $(z \neq 0)$

(ii) Show that ~~if~~ u and v satisfy the Cauchy-Riemann equation everywhere for (a), and for all $z \neq 0$ in (b)

(iii) Write the function $f(z) = |z|$ in the form $u(x,y) + iv(x,y)$. Using the Cauchy-Riemann equations, decide whether there are any points in $\mathbb{C}$ at which f is differentiable.

3. (3) Suppose $f(z) = u + iv$. Suppose we know that $u(x,y) = x^5 - 10x^3y^2 + 5xy^4$. By using the Cauchy-Riemann equations, find all the possible forms of $v(x,y)$.

Hint: $v = 5x^4y - 10x^2y^3 + y^5 + C$, $C \in \mathbb{R}$.

4. (4) Suppose that $u(x,y) = x^3 - kxy^2 + 12xy - 12x$ for some constant $k \in \mathbb{C}$. Find all values of k for which u is the real part of a holomorphic function $f: \mathbb{C} \to \mathbb{C}$.

Hint: $\Delta u = 0$.

5. (2) Show that the only holomorphic function $f: \mathbb{C} \to \mathbb{C}$ of the form $f(x+iy) = u(x) + iv(y)$ is given by $f(z) = \lambda z + a$ for some $\lambda \in \mathbb{R}$ and $a \in \mathbb{C}$.

Hint; $\Delta u = 0$, $\Delta v = 0$, with $\Delta := \partial_{xx} + \partial_{yy}$

6. (1) Suppose that $f(z) = u(x,y) + iv(x,y)$, $f: \mathbb{C} \to \mathbb{C}$, is a holomorphic function and that

$$2u(x,y) + v(x,y) = 5 \quad \text{for all } z = x + iy \in \mathbb{C}.$$

Show that f is constant.

Hint: define $h(z) = (2-i) \cdot f(z)$